MA 3053 Practice Exam 2 Solutions

1.) consider the following truth table:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
0	0	1	0	1
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

□

2.) Consider the following:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
0	0	1	1	1
1	0	0	0	0
0	1	1	1	1
1	1	1	0	1

can see the 3rd and 5th column are the same $\therefore (P \Rightarrow Q) \equiv (\neg P \vee Q)$ □

3.) Suppose $\sqrt{14}$ is rat'l. Then $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ w/ no common factors s.t. $\sqrt{14} = \frac{p}{q}$ $\Rightarrow p^2 = 14q^2$ $\Rightarrow p^2 \equiv_{14} 0$. Since modular arith. preserves multip. $\Rightarrow p \equiv_{14} 0$. $\therefore \exists\, K \in \mathbb{Z}$ s.t. $p = 14K$ $\Rightarrow (14K)^2 = 14q^2 \Rightarrow q^2 = 14K^2$ $\therefore q^2 \equiv_{14} 0$. Similarly $q \equiv_{14} 0$ $\therefore$ q has a factor of 14. contradiction. □

4.) Consider $\sqrt{2}$. Know $\sqrt{2}$ is irrational. Then $\sqrt{2}^{\sqrt{2}}$ is either irrational or rational. If it's rat'l, then done. otherwise $\sqrt{2}^{\sqrt{2}}$ is irrat'l. so $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$. □

5.) By the MVT $\exists\, c \in (-6, 6)$ s.t. $f(6) - f(-6) = f'(c)(6 - -6) \Rightarrow f(6) - f(-6) = 26 f'(c)$
since f is odd, $f(-6) = -f(6)$ $\therefore$ $2f(6) = 26f'(c)$. since $6 > 0 \Rightarrow f'(c) = \frac{f(6)}{6}$ □

6.) Suppose that either $a < \sqrt{c}$ and $b < \sqrt{c}$ (1) or $a > \sqrt{c}$ and $b > \sqrt{c}$ (2). If (1) then $ab < \sqrt{c} \cdot \sqrt{c} = c \Rightarrow ab < c$ contradicts $ab = c$. If (2) then $ab > \sqrt{c} \cdot \sqrt{c} = c \Rightarrow ab > c$. Also contradicts $ab = c$.
$\therefore$ must have one of $a \le \sqrt{c}$ or $b \le \sqrt{c}$. □

7.) <u>Base case</u>: $n = 0$: $LHS = \sum_{k=0}^{0} k^2 = 0^2 = 0$. $RHS = \frac{0(0+1)(2 \cdot 0 + 1)}{6} = 0$ $\therefore RHS = LHS$.

<u>Inductive step</u>: Suppose the equ. holds for some $N \in \mathbb{N}$. Consider $\sum_{k=0}^{N+1} k^2 = (N+1)^2 + \sum_{k=0}^{N} k^2$ so by assumption

$$= (N+1)^2 + \frac{N(N+1)(2N+1)}{6} = (N+1)\left(\frac{6N+6}{6} + \frac{2N^2+N}{6}\right) = \frac{N+1}{6}(2N^2 + 7N + 6) = \frac{(N+1)(N+2)(2N+3)}{6}$$

$\therefore$ by the principle of induction the result holds. □

8.) (a) Know $0 + 0 = 0$ $\therefore$ $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$.

(b) <u>Base case</u>: $n = 1$ $LHS = f(1)$, $RHS = 1 \cdot f(1) = f(1)$. so $RHS = LHS$. <u>Inductive step</u>: suppose the equat. holds for some $K \in \mathbb{N}$. Consider $f(K+1) = f(K) + f(1) = K f(1) + f(1)$ by assumpt. $= (K+1) f(1)$. $\therefore$ the result holds by principle of induction. □

9.) let $a_1, \dots, a_n$ be rat'l #s. $\therefore \exists\, p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{Z}$ $q_j \neq 0$ $j = 1, \dots, n$ s.t. $a_j = \frac{p_j}{q_j}$ $j = 1, \dots, n$
then $a_1 a_2 \cdots a_n = \frac{p_1}{q_1} \cdots \frac{p_n}{q_n} = \frac{p_1 \cdot p_2 \cdots p_n}{q_1 \cdots q_n}$. since $p_1 \cdots p_n \in \mathbb{Z}$ and $q_1 \cdots q_n \in \mathbb{Z}$ and $\neq 0$
$\Rightarrow a_1 \cdots a_n \in \mathbb{Q}$. Now consider $\sqrt{2}$. it's irrat'l but $\sqrt{2} \cdot \sqrt{2} = 1 \in \mathbb{Q}$. □